

SELF-DUAL WULFF SHAPES AND SPHERICAL CONVEX BODIES OF CONSTANT WIDTH $\pi/2$

HUHE HAN AND TAKASHI NISHIMURA

ABSTRACT. For any Wulff shape, its dual Wulff shape is naturally defined. A self-dual Wulff shape is a Wulff shape equaling its dual Wulff shape exactly. In this paper, it is shown that a Wulff shape is self-dual if and only if the spherical convex body induced by it is of constant width $\pi/2$.

1. INTRODUCTION

For a positive integer n , let S^n be the unit sphere in \mathbb{R}^{n+1} . Let \mathbb{R}_+ be the set consisting of positive real numbers. For any continuous function $\gamma : S^n \rightarrow \mathbb{R}_+$ and any $\theta \in S^n$, let $\Gamma_{\gamma, \theta}$ be the set consisting of $x \in \mathbb{R}^{n+1}$ such that $x \cdot \theta \leq \gamma(\theta)$, where the dot in the center stands for the scalar product of two vectors $x, \theta \in \mathbb{R}^{n+1}$. Then, the *Wulff shape* associated with the support function γ is the following set \mathcal{W}_γ :

$$\mathcal{W}_\gamma = \bigcap_{\theta \in S^n} \Gamma_{\gamma, \theta}.$$

A Wulff shape \mathcal{W}_γ was firstly introduced by G. Wulff in [6] as a geometric model of a crystal at equilibrium. By definition, any Wulff shape is a convex body in \mathbb{R}^{n+1} containing the origin as an interior point. Conversely, it has been known that for any convex body W in \mathbb{R}^{n+1} such that $\text{int}(W)$ contains the origin where $\text{int}(W)$ stands for the set of interior points of W , there exists a continuous function $\gamma : S^n \rightarrow \mathbb{R}_+$ such that $W = \mathcal{W}_\gamma$ ([5]). By using the polar plot expression of elements of $\mathbb{R}^{n+1} - \{0\}$, $S^n \times \mathbb{R}_+$ may be naturally identified with $\mathbb{R}^{n+1} - \{0\}$. Under this identification, for any Wulff shape \mathcal{W}_γ and any $\theta \in S^n$, the intersection $\partial\mathcal{W}_\gamma \cap L_\theta$ is exactly one point (denoted by $(\theta, w(\theta))$), where $\partial\mathcal{W}_\gamma$ is the boundary of \mathcal{W}_γ and L_θ is the half line $L_\theta = \{(\theta, r) \mid r \in \mathbb{R}_+\}$. For a Wulff shape \mathcal{W}_γ , let $\bar{\gamma} : S^n \rightarrow \mathbb{R}_+$ be the continuous function defined by $\bar{\gamma}(\theta) = \frac{1}{w(-\theta)}$. Then, the Wulff shape $\mathcal{W}_{\bar{\gamma}}$ is called the *dual Wulff shape* of \mathcal{W}_γ and is denoted by \mathcal{DW}_γ . For any Wulff shape \mathcal{W}_γ , there is a characterization of the dual Wulff shape of \mathcal{W}_γ . The graph of a continuous function $\gamma : S^n \rightarrow \mathbb{R}_+$ is denoted by $\text{graph}(\gamma)$. Let $\text{inv} : \mathbb{R}^{n+1} - \{0\} \rightarrow \mathbb{R}^{n+1}$ be the inversion of $\mathbb{R}^{n+1} - \{0\}$ defined by $\text{inv}(\theta, r) = (-\theta, \frac{1}{r})$. Then, for any continuous function $\gamma : S^n \rightarrow \mathbb{R}_+$, \mathcal{DW}_γ is exactly the convex hull of $\text{inv}(\text{graph}(\gamma))$. By this characterization, it is clear that $\mathcal{DDW}_\gamma = \mathcal{W}_\gamma$ for any \mathcal{W}_γ when $\text{inv}(\text{graph}(\gamma))$ is the boundary of the convex hull of $\text{inv}(\text{graph}(\gamma))$. A Wulff shape \mathcal{W}_γ is said to be *self-dual* if the equality $\mathcal{W}_\gamma = \mathcal{DW}_\gamma$ holds.

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In this paper, a simple and useful characterization for a self-dual Wulff shape in \mathbb{R}^{n+1} is given. In order to state our characterization, several notions in S^{n+1} are defined. For any point P of S^{n+1} , let $H(P)$ be the hemisphere centered at P , namely $H(P)$ is the subset of S^{n+1} consisting of $Q \in S^{n+1}$ satisfying $P \cdot Q \geq 0$, where the dot in the center stands for the scalar product of two vectors $P, Q \in \mathbb{R}^{n+2}$. A subset \widetilde{W} of S^{n+1} is said to be *hemispherical* if there exists a point $P \in S^{n+1}$ such that $\widetilde{W} \cap H(P) = \emptyset$. A hemispherical subset $\widetilde{W} \subset S^{n+1}$ is said to be *spherical convex* if for any $P, Q \in \widetilde{W}$ the following arc PQ is contained in \widetilde{W} :

$$PQ = \left\{ \frac{(1-t)P + tQ}{\|(1-t)P + tQ\|} \mid t \in [0, 1] \right\}.$$

A hemispherical subset \widetilde{W} is called a *spherical convex body* if it is closed, spherical convex and has an interior point. A hemisphere $H(P)$ is said to *support a spherical convex body* \widetilde{W} if both $\widetilde{W} \subset H(P)$ and $\partial\widetilde{W} \cap \partial H(P) \neq \emptyset$ hold. For a spherical convex body \widetilde{W} and a hemisphere $H(P)$ supporting \widetilde{W} , following [2, 3], the width of \widetilde{W} determined by $H(P)$ is defined as follows. For any two $P, Q \in S^{n+1}$ ($P \neq \pm Q$), the intersection $H(P) \cap H(Q)$ is called a *lune* of S^{n+1} . The *thickness of the lune* $H(P) \cap H(Q)$, denoted by $\Delta(H(P) \cap H(Q))$, is the real number $\pi - |PQ|$, where $|PQ|$ stands for the length of the arc PQ . For a spherical convex body \widetilde{W} and a hemisphere $H(P)$ supporting \widetilde{W} , the *width of \widetilde{W} determined by $H(P)$* , denoted by $\text{width}_{H(P)}\widetilde{W}$, is the minimum of the following set:

$$\left\{ \Delta(H(P) \cap H(Q)) \mid \widetilde{W} \subset H(P) \cap H(Q), H(Q) \text{ supports } \widetilde{W} \right\}.$$

For any $\rho \in \mathbb{R}_+$ less than π , a spherical convex body $\widetilde{W} \subset S^{n+1}$ is said to be of *constant width ρ* if $\text{width}_{H(P)}\widetilde{W} = \rho$ for any $H(P)$ supporting \widetilde{W} .

Let $Id : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \{1\} \subset \mathbb{R}^{n+2}$, $N \in S^{n+1}$ and $\alpha_N : S^{n+1} - H(-N) \rightarrow \mathbb{R}^{n+1} \times \{1\} \subset \mathbb{R}^{n+2}$ be the mapping defined by $Id(x) = (x, 1)$, the point $(0, \dots, 0, 1) \in S^{n+1}$ and the central projection defined as follows respectively.

$$\alpha_N(P_1, \dots, P_{n+1}, P_{n+2}) = \left(\frac{P_1}{P_{n+2}}, \dots, \frac{P_{n+1}}{P_{n+2}}, 1 \right) \\ (\forall (P_1, \dots, P_{n+1}, P_{n+2}) \in S^{n+1} - H(-N)).$$

Then, for any Wulff shape \mathcal{W}_γ , it is clear that $\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ is a spherical convex body. The set $\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ is called the *spherical convex body induced by \mathcal{W}_γ* .

Theorem 1. *Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a continuous function. Then, the Wulff shape \mathcal{W}_γ is self-dual if and only if the spherical convex body induced by \mathcal{W}_γ is of constant width $\pi/2$.*

The unit disc $D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$ of \mathbb{R}^{n+1} is clearly self-dual. Let R be a rotation of \mathbb{R}^{n+2} about an n dimensional linear subspace with a small angle. Then, since the property of constant width is an invariant property by R , by Theorem 1, $Id^{-1} \circ \alpha_N(R(\alpha_N^{-1} \circ Id(D^{n+1})))$ is self-dual as well (see Figure 1). Moreover, let $\widetilde{\Delta}$ be a spherical triangle of constant width $\frac{\pi}{2}$ in S^2 containing N as an interior point. Then, by Theorem 1, not only $Id^{-1} \circ \alpha_N(\widetilde{\Delta})$ itself, but also any $Id^{-1} \circ \alpha_N(R(\widetilde{\Delta}))$ is self-dual (see Figure 2). For more consideration on simple, explicit examples, see Section 4.

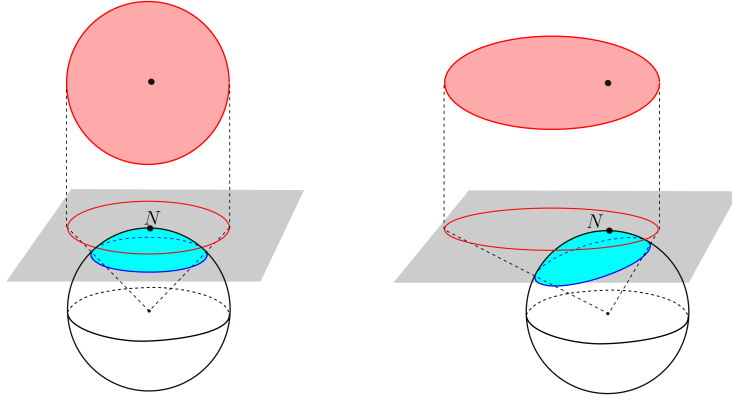


FIGURE 1. Self-dual Wulff shapes include central projections of spherical caps of width $\pi/2$.

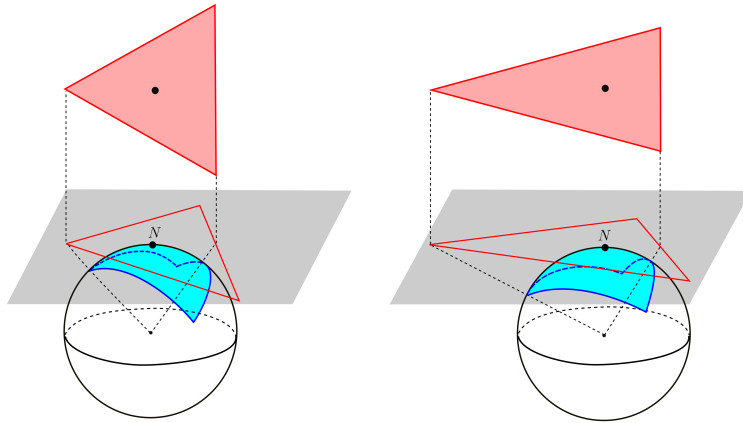


FIGURE 2. Self-dual Wulff shapes include triangles which are central projections of constant-width spherical triangles of width $\pi/2$.

On the other hand, any Reuleaux triangle in \mathbb{R}^2 containing the origin as an interior point (see Figure 3) is not a self-dual Wulff shape, although it is a Wulff shape of constant width in \mathbb{R}^2 . This is because any Reuleaux triangle is strictly convex, and thus the boundary of it must be smooth by [1] if it is self-dual. However, there are three non-smooth points for any Reuleaux triangle in \mathbb{R}^2 . By Theorem 1, its spherical convex body is not of constant width $\pi/2$.

In Section 2, preliminaries for the proof of Theorem 1 are given. The proof of Theorem 1 is given in Section 3. Finally, more consideration on simple, explicit examples is given.

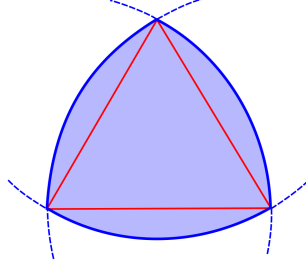


FIGURE 3. Reuleaux triangle.

2. PRELIMINARIES

The following two theorems given in [2] are keys for the proof of Theorem 1.

Theorem 2 ([2]). *Let $\widetilde{W} \subset S^{n+1}$ be a spherical convex body and let $H(P)$ be a hemisphere which supports \widetilde{W} .*

- (1) *If $P \notin \widetilde{W}$, then there exists a unique hemisphere $H(Q)$ supporting \widetilde{W} such that the lune $H(P) \cap H(Q)$ contains \widetilde{W} and has thickness $\text{width}_P(\widetilde{W})$. This hemisphere supports \widetilde{W} at the point R at which the largest ball $B(P, r)$ touches \widetilde{W} from outside. We have $\Delta(H(P) \cap H(Q)) = \frac{\pi}{2} - r$.*
- (2) *If $P \in \partial\widetilde{W}$, then there exists at least one hemisphere $H(Q)$ supporting \widetilde{W} such that $H(P) \cap H(Q)$ is a lune containing \widetilde{W} which has thickness $\text{width}_P(\widetilde{W})$. This hemisphere supports \widetilde{W} at $R = P$. We have $\Delta(H(P) \cap H(Q)) = \frac{\pi}{2}$.*
- (3) *If $P \in \text{int}(\widetilde{W})$, then there exists at least one hemisphere $H(Q)$ supporting \widetilde{W} such that $H(P) \cap H(Q)$ is a lune containing \widetilde{W} which has thickness $\text{width}_P(\widetilde{W})$. Every such $H(Q)$ supports \widetilde{W} at exactly one point $R \in \partial\widetilde{W} \cap B(P, r)$, where $B(P, r)$ denotes the largest ball with center P contained in \widetilde{W} , and for every such R this hemisphere $H(Q)$, denoted $H_R(Q)$, is unique. For every R we have $\Delta(H(P) \cap H(Q)) = \frac{\pi}{2} + r$.*

Definition 1 ([2]). Let $\widetilde{W} \subset S^{n+1}$ be a spherical convex body. Then, the following real number is called the *diameter* of \widetilde{W} and is denoted by $\text{diam}(\widetilde{W})$.

$$\max \left\{ |PQ| \mid P, Q \in \widetilde{W} \right\}.$$

Theorem 3 ([2]). *Let $\widetilde{W} \subset S^{n+1}$ be a spherical convex body. Suppose that $\text{diam}(\widetilde{W}) \leq \frac{\pi}{2}$. Then, the following holds:*

$$\begin{aligned} \text{diam}(\widetilde{W}) = \\ \max \left\{ \text{width}_{H(P)}(\widetilde{W}) \mid H(P) \text{ is a supporting hemisphere of } \widetilde{W} \right\}. \end{aligned}$$

Definition 2 ([4]). For any hemispherical subset \widetilde{W} of S^{n+1} , the following set (denoted by $\text{s-conv}(\widetilde{W})$) is called the *spherical convex hull* of \widetilde{W} :

$$\text{s-conv}(\widetilde{W}) = \left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \mid P_i \in \widetilde{W}, \sum_{i=1}^k t_i = 1, t_i \geq 0, k \in \mathbb{N} \right\}.$$

It is clear that $\text{s-conv}(\widetilde{W}) = \widetilde{W}$ if \widetilde{W} is spherical convex. More generally, we have the following:

Lemma 2.1 ([4]). *Let \widetilde{W} be a hemispherical subset of S^{n+1} . Then, the spherical convex hull of \widetilde{W} is the smallest spherical convex set containing \widetilde{W} .*

Definition 3 ([4]). For any subset \widetilde{W} of S^{n+1} , the set

$$\bigcap_{P \in \widetilde{W}} H(P)$$

is called the *spherical polar set* of \widetilde{W} and is denoted by \widetilde{W}° .

For the spherical polar sets, the following lemma is fundamental.

Lemma 2.2 ([4]). *For any non-empty closed hemispherical subset $\widetilde{W} \subset S^{n+1}$, the equality $\text{s-conv}(\widetilde{W}) = \left(\text{s-conv}(\widetilde{W}) \right)^{\circ\circ}$ holds.*

3. PROOF OF THEOREM 1

By the definition of the dual Wulff shape \mathcal{DW}_γ for a given Wulff shape \mathcal{W}_γ , it is sufficient to show the following:

Proposition 1. *Set $\widetilde{W} = \alpha_N^{-1} \circ \text{Id}(\mathcal{W}_\gamma)$. Then, $\widetilde{W} = \widetilde{W}^\circ$ if and only if \widetilde{W} is of constant width $\pi/2$.*

3.1. Proof of the “if” part of Proposition 1. In this subsection, we show that $\widetilde{W} = \widetilde{W}^\circ$ under the assumption that \widetilde{W} is of constant width $\frac{\pi}{2}$. We first show the inclusion $\widetilde{W} \subset \widetilde{W}^\circ$. Let P_1, Q_1 be two points of $\partial\widetilde{W}$ such that $|P_1 Q_1| = \text{diam}(\widetilde{W})$. Set $P_1 = (r\theta, x_{n+2})$ ($0 < r, x_{n+2} < 1, \theta \in S^n$). Since \widetilde{W} is a spherical convex body, for the $\theta \in S^n$, there exists the unique real number t ($0 < t < 1$) such that $H\left(\frac{t\theta + (1-t)N}{\|t\theta + (1-t)N\|}\right)$ supports \widetilde{W} . For the t , set $P = \frac{t\theta + (1-t)N}{\|t\theta + (1-t)N\|}$. Then, since we have assumed that \widetilde{W} is of constant width $\frac{\pi}{2}$, by Theorem 2, we have that $P \in \partial\widetilde{W}$. This implies $P_1 = P$ and hemisphere $H(P_1)$ supports \widetilde{W} . Since $Q_1 \in \widetilde{W} \subset H(P_1)$, we have the following,

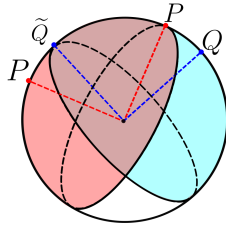
$$\text{diam}(\widetilde{W}) = |P_1 Q_1| \leq \frac{\pi}{2}.$$

Let R be an arbitrary point of \widetilde{W} . Since $\text{diam}(\widetilde{W}) \leq \frac{\pi}{2}$, the following holds,

$$R \in \bigcap_{\tilde{R} \in \widetilde{W}} H(\tilde{R}) = \widetilde{W}^\circ.$$

Therefore, we have $\widetilde{W} \subset \widetilde{W}^\circ$.

Next we show the converse inclusion $\widetilde{W}^\circ \subset \widetilde{W}$. Suppose that there exists a point $P \in \widetilde{W}^\circ$ such that $P \notin \widetilde{W}$. By Lemma 2.2, it follows that $P \notin \widetilde{W} = \bigcap_{Q \in \widetilde{W}^\circ} H(Q)$.

FIGURE 4. $|PQ| > \frac{\pi}{2}$.

This implies that there exist two points P and Q of \widetilde{W}° such that $|PQ| > \frac{\pi}{2}$. For these two points $P, Q \in \widetilde{W}^\circ$, set $\tilde{P} = PQ \cap \partial H(P)$, $\tilde{Q} = PQ \cap \partial H(Q)$ (see Figure 4). Then we have the following,

$$\pi = |P\tilde{P}| + |\tilde{Q}Q| = |P\tilde{Q}| + |\tilde{Q}\tilde{P}| + |\tilde{Q}\tilde{P}| + |\tilde{P}Q| = |PQ| + |\tilde{P}\tilde{Q}|.$$

By the assumption, it follows that $|\tilde{P}\tilde{Q}| < \frac{\pi}{2}$. Let $H(\tilde{R})$ be a supporting hemisphere of \widetilde{W} whose boundary is perpendicular to the arc PQ at the intersecting point. Then, the following holds:

$$\text{width}_{H(\tilde{R})}(\widetilde{W}) \leq |\tilde{P}\tilde{Q}| < \frac{\pi}{2}.$$

This contradicts the assumption that \widetilde{W} is of constant width $\frac{\pi}{2}$. Therefore, it follows that $\widetilde{W}^\circ \subset \widetilde{W}$. \square

3.2. Proof of the "only if" part of Proposition 1. In this subsection, we show that \widetilde{W} is of constant width $\frac{\pi}{2}$ under the assumption that $\widetilde{W} = \widetilde{W}^\circ$. Suppose that there exists a hemisphere $H(P)$ supporting \widetilde{W} such that $\text{width}_{H(P)}(\widetilde{W}) > \frac{\pi}{2}$. By Theorem 3, it follows that $\text{diam}(\widetilde{W}) \geq \text{width}_{H(P)}(\widetilde{W}) > \frac{\pi}{2}$. This implies that there exist two points $P, Q \in \widetilde{W}$ such that $P \notin H(Q)$. Then, we have the following:

$$P \notin \bigcap_{Q \in \widetilde{W}} H(Q) = \widetilde{W}^\circ.$$

This contradicts the assumption $\widetilde{W} = \widetilde{W}^\circ$.

Suppose that there exists a hemisphere $H(P)$ supporting \widetilde{W} such that the following holds:

$$\text{width}_{H(P)}(\widetilde{W}) < \frac{\pi}{2}.$$

Then, there exists a hemisphere $H(Q)$ supporting \widetilde{W} such that the following holds:

$$\Delta(H(P) \cap H(Q)) = \text{width}_{H(P)}(\widetilde{W}) < \frac{\pi}{2}.$$

Since the thickness $\Delta(H(P) \cap H(Q)) = \pi - |PQ|$, we have the following:

$$|PQ| > \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

On the other hand, since \widetilde{W} is a subset of $H(P)$ (resp. $H(Q)$), it follows that $P \in \widetilde{W}^\circ = \widetilde{W}$ (resp. $Q \in \widetilde{W}^\circ = \widetilde{W}$). This implies $\text{diam}(\widetilde{W}) \geq |PQ| > \frac{\pi}{2}$. Thus, we have a contradiction. \square

Remark. By the proof of Proposition 1, it can be seen that Proposition 1 holds for any spherical convex body \widetilde{W} . Namely, we have the following:

Proposition 2. *Let $\widetilde{W} \subset S^{n+1}$ be a spherical convex body. Then, $\widetilde{W} = \widetilde{W}^\circ$ if and only if \widetilde{W} is of constant width $\pi/2$.*

4. MORE ON SIMPLE, EXPLICIT EXAMPLES

4.1. Centrally symmetric self-dual Wulff shapes. In this subsection, we determine centrally symmetric Wulff shapes. Here, a convex body $W \subset \mathbb{R}^{n+1}$ is said to be *centrally symmetric* if $x \in W$ implies $-x \in W$.

Proposition 3. *Let $W \subset \mathbb{R}^{n+1}$ be a self-dual Wulff shape. Then, W is centrally symmetric if and only if W is the unit disc D^{n+1} .*

Proof. The “if” part is clear. We show the “only if” part. Suppose that there exists a centrally symmetric self-dual Wulff shape W which is not the unit disc D^{n+1} . Then one of the following holds.

- (1) There exist a point $p \in W$ such that $\|p\| > 1$.
- (2) The inequality $\|p\| \leq 1$ holds for any point p of W and there exists a point $q \in \partial W$ such that $\|q\| < 1$.

Here, $\|x\|$ is the distance from the origin to the point $x \in \mathbb{R}^{n+1}$.

Suppose that (1) holds. Then, since W is centrally symmetric, it follows that $-p \in W$. Set $\tilde{p} = \frac{p}{\|p\|} \in S^n$. For any point $x \in \mathbb{R}^{n+1}$, set $X_+ = \alpha_N^{-1} \circ Id(x)$ and $X_- = \alpha_N^{-1} \circ Id(-x)$. Notice that $P_- \in \widetilde{W} = \alpha_N^{-1} \circ Id(W)$. Since the distance $|\tilde{P}_+ \tilde{P}_-|$ is equal to $\frac{\pi}{2}$, we have the following:

$$\frac{\pi}{2} = |\tilde{P}_+ \tilde{P}_-| < |P_+ P_-|.$$

This implies $P_+ \notin H(P_-)$. Thus, it follows that

$$P_+ \notin \bigcap_{Q \in \widetilde{W}} H(Q) = \widetilde{W}^\circ.$$

On the other hands, since W is a self-dual Wulff shape and $p \in W$, we have that $P_+ \in \widetilde{W} = \widetilde{W}^\circ$. Therefore, we have a contradiction.

Next, suppose that (2) holds. Since there exists a point $q \in \partial W$ such that $\|q\| < 1$, it follows that the point $\frac{q}{\|q\|} \in S^n$ does not belong to W . Set $\tilde{q} = \frac{q}{\|q\|}$. Then, since W is a self-dual Wulff shape, it follows that $\tilde{Q}_+ \notin \widetilde{W} = \widetilde{W}^\circ$. On the other hands, by the assumption (2), the following holds.

$$\widetilde{W} \subset \alpha_N^{-1} \circ Id(D^{n+1}) \subset H(\tilde{Q}_+).$$

Thus, \tilde{Q}_+ is a point of \widetilde{W}° and we have a contradiction. \square

4.2. Self-dual Wulff shapes of polytope type. A Wulff shape is said to be of *polytope type* if there exist finitely many points $P_1, \dots, P_k \in S^{n+1}$ such that $\widetilde{W} = \bigcap_{i=1}^k H(P_i)$, where \widetilde{W} is a spherical convex body induced by W and $k \geq n+2 \in \mathbb{N}$. For crystallines, we have the following proposition:

Proposition 4. *Let $W \subset \mathbb{R}^{n+1}$ be a Wulff shape of polytope type and let \widetilde{W} be a spherical convex body induced by W . Set $\widetilde{W} = \bigcap_{i=1}^k H(P_i) \subset S^{n+1}$. Then, W is a self-dual Wulff shape if and only if P_i is a vertex of \widetilde{W} for any i ($1 \leq i \leq k$).*

Proof: For the proof, the following lemma is needed.

Lemma 4.1 (Maehara's Lemma [4]). *For any hemispherical finite subset $X = \{P_1, \dots, P_k\}$, the following holds:*

$$\left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \mid P_i \in X, \sum_{i=1}^k t_i = 1, t_i \geq 0 \right\}^\circ = \bigcap_{i=1}^k H(P_i).$$

Proof of the "only if" part: Let W be a self-dual Wulff shape of polytope type. Then, by Maehara's Lemma, we have the following equality:

$$\widetilde{W} = \bigcap_{i=1}^k H(P_i) = \left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \mid P_i \in X, \sum_{i=1}^k t_i = 1, t_i \geq 0 \right\}^\circ.$$

Then, by Lemma 2.2, the following holds:

$$\widetilde{W}^\circ = \left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \mid P_i \in X, \sum_{i=1}^k t_i = 1, t_i \geq 0 \right\}.$$

Since W is a self-dual Wulff shape, it follows that $\widetilde{W} = \widetilde{W}^\circ$. Hence, P_i is a vertex of \widetilde{W} for any i ($1 \leq i \leq 2m+1$).

Proof of the "if" part: Since P_i is a vertex of \widetilde{W} , we have the following:

$$\widetilde{W} = \left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \mid P_i \in X, \sum_{i=1}^k t_i = 1, t_i \geq 0 \right\}.$$

Thus, by Maehara's Lemma, we have the following:

$$\widetilde{W}^\circ = \left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \mid P_i \in X, \sum_{i=1}^k t_i = 1, t_i \geq 0 \right\}^\circ = \bigcap_{i=1}^k H(P_i) = \widetilde{W}.$$

Therefore, W is a self-dual Wulff shape. \square

4.3. When is the dual Wulff shape congruent to the original Wulff shape ? Finally, as a generalized problem of characterization of self-dual Wulff shapes, we pose the following:

Problem 1. *Under what conditions is the dual Wulff shape merely congruent to the original Wulff shape ?*

We have partial results to this problem as follows:

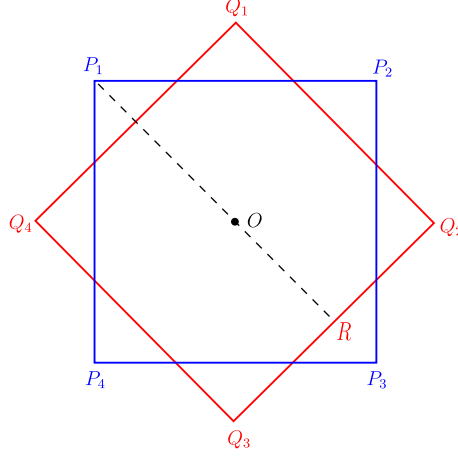


FIGURE 5. Square $P_1P_2P_3P_4$ and its dual square $Q_1Q_2Q_3Q_4$.
 $P_1O = a_4$, $RO = \frac{1}{a_4}$, where $a_4^2 = \sqrt{2}$.

Example 1. Let X_{2m} be a regular polygon with $2m$ vertices in the plane where $m \geq 2$. Denote the half of the length of its diagonal by a_{2m} . Suppose that the center of X_{2m} is the origin and a_{2m} satisfies the following equation:

$$(*) \quad \sin\left(\frac{\pi - \frac{2\pi}{2m}}{2}\right) = \frac{\frac{1}{a_{2m}}}{a_{2m}}.$$

Then, $X_{2m} \neq DX_{2m}$ but DX_{2m} is congruent to X_{2m} .

For instance, consider a square $P_1P_2P_3P_4 \subset \mathbb{R}^2$ such that the origin is its center and the length of its edge is $\frac{2}{a_4}$, where $a_4^2 = \sqrt{2}$. Let $Q_1Q_2Q_3Q_4 \subset \mathbb{R}^2$ be the dual Wulff shape of $P_1P_2P_3P_4$. Then, $P_1P_2P_3P_4 \neq Q_1Q_2Q_3Q_4$ (see Figure 5). And, it is easy to see that $Q_1Q_2Q_3Q_4$ is also a square with properties that the origin is its center and the length of its edge is $\frac{2}{a_4}$. Thus, $Q_1Q_2Q_3Q_4$ is congruent to $P_1P_2P_3P_4$.

It is not difficult to obtain the equation $(*)$ for a_{2m} of general $2m$ -gon X_{2m} .

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GRADUATE SCHOOL OF ENVIRONMENT AND INFORMATION SCIENCES, YOKOHAMA NATIONAL UNIVERSITY, YOKOHAMA 240-8501, JAPAN

E-mail address: han-huhe-bx@ynu.jp

RESEARCH INSTITUTE OF ENVIRONMENT AND INFORMATION SCIENCES, YOKOHAMA NATIONAL UNIVERSITY, YOKOHAMA 240-8501, JAPAN

E-mail address: nishimura-takashi-yx@ynu.jp